ity, able to meet the public, and compose a decent letter. He must have the mathematical ability to enable him to pass his actuarial examinations and qualify as an actuary, but so far as his practical work is concerned, in the training of his actuarial department for the regular routine work of the office he can forget if he will practically all of his higher mathematics. When an actuary is called upon, as is frequently the case, to take his part in the executive work of the company, it is necessary that he should have some knowledge of life insurance law, the principles of accounting, and a general knowledge of investments and finance. From what I have said I believe that you will agree with me when I say that an actuary is not necessarily an expert mathematician, and that an expert mathematician will not necessarily make a good actuary.

---

# NOTE ON THE ROOTS OF THE DERIVATIVE OF A POLYNOMIAL.

By W. H. ECHOLS, University of Virginia.

(Read before the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America, May 15, 1920.)

The problem of determining regions to which are confined the roots of the derivatives of functions has received a great deal of attention. Lucas established the fact[1], by a mechanical proof, that the roots of the derivative of any polynomial are confined to the smallest convex region enclosing the roots of the polynomial. Maxime Bôcher gave a geometrical demonstration of this.[2] He noted that the roots of the derivative of any cubic equation are the foci of the ellipse tangent at their mid-points to the sides of the triangle of the roots of the cubic; and remarked that it might be possible to associate the roots of the derivative of any polynomial with the foci of a higher plane curve. B. Z. Linfield gave a beautiful demonstration, in a paper read at the meeting, in St. Louis last December, of the American Mathematical Society (*Bulletin*, p. 264), that the roots of the derivative of any polynomial of degree $n$ were the foci of a curve of class $n-1$ touching at their mid-points the segments joining, two and two, the roots of the polynomial. Recently J. L. W. V. Jensen stated,[3] without demonstration,

[1] F. Lucas, "Géométrie des polynômes," *Journal de l'Ecole Polytechnique*, 1879, cahier 46, tome 28, pp. 1–33.—Editor.

[2] M. Bôcher, "Some propositions concerning the geometric representation of imaginaries," *Annals of Mathematics*, March, 1893, vol. 7, pp. 70–72.—Editor.

[3] In a letter to the writer Dr. J. L. Walsh, of Harvard, says: "Jensen originally stated his theorem without proof, in *Acta Mathematica*, vol. 36 (1912), p. 190. Apparently the theorem remained unnoticed until Professor D. R. Curtiss called attention to it in a paper presented at a meeting of the American Mathematical Society; his abstract was published in the current volume of the *Bulletin*, pp. 61–62. Professor Curtiss presented further results in April (see the June *Bulletin*, p. 392). I also have some further results which I presented to the Society in December; the abstract of which is in the March *Bulletin*, p. 259. My paper has been accepted for publication in the *Annals of Mathematics*; it contains a proof of Jensen's theorem based on mechanical considerations. Your own proof is surely different from mine (in form but not in substance). I do not know whether it is different from Curtiss's. I was not aware an algebraic proof could be given so simply. No proof of Jensen's theorem has yet been published."

that the complex roots of the derivative of a polynomial with real coefficients lie inside circles having the segments joining the pairs of conjugate roots of the polynomial as diameters. It is the purpose of this note to present a complete demonstration establishing the truth of this statement.

THEOREM. If $f(z)$ is a polynomial whose coefficients are real numbers, then the imaginary roots of the derivative $f'(z)$ lie on or within circles whose diameters are the segments joining the pairs of conjugate imaginary roots of $f(z)$.

*Proof.* Let

$$f(z) \equiv L_1^{p_1}L_2^{p_2}\cdots L_m^{p_m}\cdot Q_1^{q_1}Q_2^{q_2}\cdots Q_n^{q_n},$$

wherein $p_r$ and $q_r$ are positive numbers, and

$$L_r \equiv z - a_r, \qquad Q_r \equiv (z - b_r)^2 + c_r^2,$$

$a_r$, $b_r$, $c_r$ real numbers. Taking the logarithm and differentiating, the derivative is

$$f'(z) = f(z)\cdot\Sigma\left(\frac{p_r}{z - a_r} + 2q_r\frac{z - b_r}{Q_r}\right).$$

Realizing the denominators ($z \equiv x + iy$)

$$\frac{p_r}{z - a_r} = \frac{p_r}{x - a_r + iy} = p_r\frac{(x - a_r) - iy}{(x - a_r)^2 + y^2},$$

and

$$\frac{z - b_r}{Q_r} = (x - b_r)\frac{(x - b_r)^2 + y^2 + c_r^2}{D_r} - iy\frac{(x - b_r)^2 + y^2 - c_r^2}{D_r},$$

where, for brevity,

$$D_r \equiv \{(x - b_r)^2 - y^2 + c_r^2\}^2 + 4(x - b_r)^2y^2.$$

Hence, equating to zero the real and imaginary components of the sigma factor of $f'(z)$, we have the necessary and sufficient conditions for those roots of $f'(z)$ which are not common to $f(z)$,

$$\Sigma\left\{p_r\frac{x - a_r}{(x - a_r)^2 + y^2} + 2q_r(x - b_r)\frac{(x - b_r)^2 + y^2 + c_r^2}{D_r}\right\} = 0,$$

$$y\Sigma\left\{\frac{p_r}{(x - a_r)^2 + y^2} + 2q_r\frac{(x - b_r)^2 + y^2 - c_r^2}{D_r}\right\} = 0.$$

The real roots are furnished by $y = 0$ and the first condition. For $y \neq 0$ it is impossible for the second condition to vanish unless the point $x$, $y$ is inside one of the circles

$$(x - b_r)^2 + y^2 = c_r^2.$$

The multiple roots of the polynomial are of course also roots of the derivative and these are on the circles. This justifies the statement.

*May* 1, 1920.